Indian Statistical Institute, Bangalore

B. Math. (hons.) First Year, First Semester

Linear Algebra I

Mid Term Examination — Date : 12 October 2022

Maximum marks: 30 — Time: 120 minutes

Answer any Five, each question carries 6 marks.

1. Prove that the number of elements in any linearly independent set is less than the number of elements in a (finite) basis of a vector space.

2. If W_1, W_2 are subspaces of a vector space, prove that $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

3. Prove that $A^2 = A$ if and only if $\mathbb{K}^m = C(A) + C(I-A)$ and $C(I-A) = N(A)$.

4. (i) Let A be a $m \times n$-matrix with rank m and B be a $r \times m$-matrix with rank r. Find the rank of BA (**Marks 4**).
   (ii) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear transformation and $v_1, v_2, \cdots, v_k$ are linearly independent vectors in $\mathbb{R}^n$. Prove that $T(v_1), T(v_2), \cdots, T(v_k)$ are also linearly independent vectors in $\mathbb{R}^n$.

5. Prove that $C(A^m) = C(A^{m+1})$ implies $C(A^n) = C(A^m)$ for all $n > m$ and there is a smallest integer m such that $C(A^m) = C(A^{m+1})$.

6. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by $T(x_1, x_2, x_3, x_4) = (x_2, 2x_3, 0, 0)$ and A be the matrix of T with respect to the standard basis. Find g-inverses of A for all possible ranks.

7. Prove that there is a g-inverse of A with rank same as that of A.